

1. Intermodular Linking Test Chapter
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Intermodular Linking Test Module 1



Example:
Insert Example Text Here

(1,1)	(1,2)	(1,3)
(2,1)	(2,2)	(2,3)
(3,1)	(3,2)	(3,3)

Intermodular Linking Test Module 2

Link to a different module (in the same collection): [\[link\]](#)

Link to a figure in a different module (in the same chapter): [\[link\]](#)

Link to an example in a different module (in the same chapter): [\[link\]](#)

Link to a table in a different module (in the same chapter): [\[link\]](#)

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Appendix A

This appendix contains outline proofs and derivations for the theorems and formulas given in early part of [Chapter: The Scaling Function and Scaling Coefficients, Wavelet and Wavelet Coefficients](#). They are not intended to be complete or formal, but they should be sufficient to understand the ideas behind why a result is true and to give some insight into its interpretation as well as to indicate assumptions and restrictions.

Proof 1 The conditions given by [\[link\]](#) and [\[link\]](#) can be derived by integrating both sides of **Equation:**

$$\varphi(x) = \sum_n h(n) \sqrt{M} \varphi(Mx - n)$$

and making the change of variables $y = Mx$

Equation:

$$\int \varphi(x) dx = \sum_n h(n) \int \sqrt{M} \varphi(Mx - n) dx$$

and noting the integral is independent of translation which gives

Equation:

$$= \sum_n h(n) \sqrt{M} \int \varphi(y) \frac{1}{M} dy.$$

With no further requirements other than $\varphi \in L^1$ to allow the sum and integral interchange and $\int \varphi(x) dx \neq 0$, this gives [\[link\]](#) as

Equation:

$$\sum_n h(n) = \sqrt{M}$$

and for $M = 2$ gives [\[link\]](#). Note this does not assume orthogonality nor any specific normalization of $\varphi(t)$ and does not even assume M is an integer.

This is the most basic necessary condition for the existence of $\varphi(t)$ and it has the fewest assumptions or restrictions.

Proof 2 The conditions in [\[link\]](#) and [\[link\]](#) are a down-sampled orthogonality of translates by M of the coefficients which results from the orthogonality of translates of the scaling function given by

Equation:

$$\int \varphi(x) \varphi(x - m) dx = E \delta(m)$$

in [\[link\]](#). The basic scaling equation [\[link\]](#) is substituted for both functions in [\[link\]](#), giving

Equation:

$$\int \left[\sum_n h(n) \sqrt{M} \varphi(Mx - n) \right] \left[\sum_k h(k) \sqrt{M} \varphi(Mx - Mm - k) \right] dx = E \delta(m)$$

which, after reordering and a change of variable $y = Mx$, gives

Equation:

$$\sum_n \sum_k h(n) h(k) \int \varphi(y - n) \varphi(y - Mm - k) dy = E \delta(m).$$

Using the orthogonality in [\[link\]](#), gives our result

Equation:

$$\sum_n h(n) h(n - Mm) = \delta(m)$$

in [\[link\]](#) and [\[link\]](#). This result requires the orthogonality condition [\[link\]](#), M must be an integer, and any non-zero normalization E may be used.

Proof 3 (Corollary 2) The result that

Equation:

$$\sum_n h(2n) = \sum_n h(2n + 1) = 1/\sqrt{2}$$

in [\[link\]](#) or, more generally

Equation:

$$\sum_n h(Mn) = \sum_n h(Mn + k) = 1/\sqrt{M}$$

is obtained by breaking [\[link\]](#), for $M = 2$ into the sum of the even and odd coefficients.

Equation:

$$\sum_n h(n) = \sum_k h(2k) + \sum_k h(2k + 1) = K_0 + K_1 = \sqrt{2}.$$

Next we use [\[link\]](#) and sum over n to give

Equation:

$$\sum_n \sum_k h(k+2n)h(k) = 1$$

which we then split into even and odd sums and reorder to give:

Equation:

mtd
mtd
mtd

Solving [\[link\]](#) and [\[link\]](#) simultaneously gives $K_0 = K_1 = 1/\sqrt{2}$ and our result [\[link\]](#) or [\[link\]](#) for $M = 2$.

If the same approach is taken with [\[link\]](#) and [\[link\]](#) for $M = 3$, we have

Equation:

$$\sum_n x(n) = \sum_n x(3n) + \sum_n x(3n+1) + \sum_n x(3n+2) = \sqrt{3}$$

which, in terms of the partial sums K_i , is

Equation:

$$\sum_n x(n) = K_0 + K_1 + K_2 = \sqrt{3}.$$

Using the orthogonality condition [\[link\]](#) as was done in [\[link\]](#) and [\[link\]](#) gives

Equation:

$$K_0^2 + K_1^2 + K_2^2 = 1.$$

Equation [\[link\]](#) and [\[link\]](#) are simultaneously true if and only if $K_0 = K_1 = K_2 = 1/\sqrt{3}$. This process is valid for any integer M and any non-zero normalization.

Proof 3 If the support of $\varphi(x)$ is $[0, N-1]$, from the basic recursion equation with support of $h(n)$ assumed as $[N_1, N_2]$ we have

Equation:

$$\varphi(x) = \sum_{n=N_1}^{N_2} h(n) \sqrt{2} \varphi(2x - n)$$

where the support of the right hand side of [\[link\]](#) is $[N_1/2, (N - 1 + N_2)/2]$. Since the support of both sides of [\[link\]](#) must be the same, the limits on the sum, or, the limits on the indices of the non zero $h(n)$ are such that $N_1 = 0$ and $N_2 = N$, therefore, the support of $h(n)$ is $[0, N - 1]$.

Proof 4 First define the autocorrelation function

Equation:

$$a(t) = \int \varphi(x) \varphi(x - t) dx$$

and the power spectrum

Equation:

$$A(\omega) = \int a(t) e^{-j\omega t} dt = \int \int \varphi(x) \varphi(x - t) dx e^{-j\omega t} dt$$

which after changing variables, $y = x - t$, and reordering operations gives

Equation:

$$A(\omega) = \int \varphi(x) e^{-j\omega x} dx \int \varphi(y) e^{j\omega y} dy$$

Equation:

$$= \Phi(\omega) \Phi(-\omega) = |\Phi(\omega)|^2$$

If we look at [\[link\]](#), as being the inverse Fourier transform of [\[link\]](#), and sample $a(t)$ at $t = k$, we have

Equation:

$$a(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi(\omega)|^2 e^{j\omega k} d\omega$$

Equation:

$$= \frac{1}{2\pi} \sum_{\ell} \int_0^{2\pi} |\Phi(\omega + 2\pi\ell)|^2 e^{j\omega k} d\omega = \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{\ell} |\Phi(\omega + 2\pi\ell)|^2 \right] e^{j\omega k} d\omega$$

but this integral is the form of an inverse discrete-time Fourier transform (DTFT) which means

Equation:

$$\sum_k a(k) e^{j\omega k} = \sum_\ell |\Phi(\omega + 2\pi\ell)|^2.$$

If the integer translates of $\varphi(t)$ are orthogonal, $a(k) = \delta(k)$ and we have our result

Equation:

$$\sum_\ell |\Phi(\omega + 2\pi\ell)|^2 = 1.$$

If the scaling function is not normalized

Equation:

$$\sum_\ell |\Phi(\omega + 2\pi\ell)|^2 = \int |\varphi(t)|^2 dt$$

which is similar to Parseval's theorem relating the energy in the frequency domain to the energy in the time domain.

Proof 6 Equation [\[link\]](#), states a very interesting property of the frequency response of an FIR filter with the scaling coefficients as filter coefficients. This result can be derived in the frequency or time domain. We will show the frequency domain argument. The scaling equation [\[link\]](#), becomes [\[link\]](#), in the frequency domain. Taking the squared magnitude of both sides of a scaled version of

Equation:

$$\Phi(\omega) = \frac{1}{\sqrt{2}} H(\omega/2) \Phi(\omega/2)$$

gives

Equation:

$$|\Phi(2\omega)|^2 = \frac{1}{2} |H(\omega)|^2 |\Phi(\omega)|^2$$

Add $k\pi$ to ω and sum over k to give for the left side of [\[link\]](#),

Equation:

$$\sum_k |\Phi(2\omega + 2\pi k)|^2 = K = 1$$

which is unity from [\[link\]](#). Summing the right side of [\[link\]](#) gives

Equation:

$$\sum_k \frac{1}{2} |H(\omega + k\pi)|^2 |\Phi(\omega + k\pi)|^2$$

Break this sum into a sum of the even and odd indexed terms.

Equation:

$$\sum_k \frac{1}{2} |H(\omega + 2\pi k)|^2 |\Phi(\omega + 2\pi k)|^2 + \sum_k \frac{1}{2} |H(\omega + (2k+1)\pi)|^2 |\Phi(\omega + (2k+1)\pi)|^2$$

Equation:

$$= \frac{1}{2} |H(\omega)|^2 \sum_k |\Phi(\omega + 2\pi k)|^2 + \frac{1}{2} |H(\omega + \pi)|^2 \sum_k |\Phi(\omega + (2k+1)\pi)|^2$$

which after using [\[link\]](#) gives

Equation:

$$= \frac{1}{2} |H(\omega)|^2 + \frac{1}{2} |H(\omega + \pi)|^2 = 1$$

which gives [\[link\]](#). This requires both the scaling and orthogonal relations but no specific normalization of $\varphi(t)$. If viewed as an FIR filter, $h(n)$ is called a quadrature mirror filter (QMF) because of the symmetry of its frequency response about π .

Proof 10 The multiresolution assumptions in [\[link\]](#) require the scaling function and wavelet satisfy [\[link\]](#), and [\[link\]](#).

Equation:

$$\varphi(t) = \sum_n h(n) \sqrt{2} \varphi(2t - n), \quad \psi(t) = \sum_n h_1(n) \sqrt{2} \varphi(2t - n)$$

and orthonormality requires

Equation:

$$\int \varphi(t) \varphi(t - k) dt = \delta(k)$$

and

Equation:

$$\int \psi(t) \varphi(t-k) dt = 0$$

for all $k \in \mathbf{Z}$. Substituting [\[link\]](#) into [\[link\]](#) gives

Equation:

$$\int \sum_n h_1(n) \sqrt{2} \varphi(2t-n) \sum_\ell h(\ell) \sqrt{2} \varphi(2t-2k-\ell) dt = 0$$

Rearranging and making a change of variables gives

Equation:

$$\sum_{n,\ell} h_1(n) h(\ell) \frac{1}{2} \int \varphi(y-n) \varphi(y-2k-\ell) dy = 0$$

Using [\[link\]](#) gives

Equation:

$$\sum_{n,\ell} h_1(n) h(\ell) \delta(n-2k-\ell) = 0$$

for all $k \in \mathbf{Z}$. Summing over ℓ gives

Equation:

$$\sum_n h_1(n) h(n-2k) = 0$$

Separating [\[link\]](#) into even and odd indices gives

Equation:

$$\sum_m h_1(2m) h(2m-2k) + \sum_\ell h_1(2\ell+1) h(2\ell+1-2k) = 0$$

which must be true for all integer k . Defining $h_e(n) = h(2n)$, $h_o(n) = h(2n+1)$ and $\tilde{g}(n) = g(-n)$ for any sequence g , this becomes

Equation:

$$h_e \star \tilde{h}_{1e} + h_o \star \tilde{h}_{1o} = 0.$$

From the orthonormality of the translates of φ and ψ one can similarly obtain the following:

Equation:

$$h_e \star \tilde{h}_e + h_o \star \tilde{h}_o = \delta.$$

Equation:

$$h_{1e} \star \tilde{h}_{1e} + h_{1o} \star \tilde{h}_{1o} = \delta.$$

This can be compactly represented as

Equation:

$$\begin{bmatrix} h_e & h_o \\ h_{1e} & h_{1o} \end{bmatrix} \star \begin{bmatrix} \tilde{h}_e & \tilde{h}_{1e} \\ \tilde{h}_o & \tilde{h}_{1o} \end{bmatrix} = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}.$$

Assuming the sequences are finite length [\[link\]](#), can be used to show that

Equation:

$$h_e \star h_{1o} - h_o \star h_{1e} = \pm \delta_k,$$

where $\delta_k(n) = \delta(n - k)$. Indeed, taking the Z-transform of [\[link\]](#), we get using the notation of [Chapter: Filter Banks and Transmultiplexers](#), $H_p(z)H_p^T(z^{-1}) = I$. Because, the filters are FIR $H_p(z)$ is a (Laurent) polynomial matrix with a polynomial matrix inverse. Therefore the determinant of $H_p(z)$ is of the form $\pm z^k$ for some integer k . This is equivalent to [\[link\]](#).

Now, convolving both sides of [\[link\]](#) by \tilde{h}_e we get

Equation:

$$\begin{aligned} \pm \tilde{h}_e \star \delta_k &= [h_e \star h_{1o} - h_o \star h_{1e}] \star \tilde{h}_e \\ &= [h_e \star \tilde{h}_e \star h_{1o} - h_{1e} \star \tilde{h}_e \star h_o] \\ &= [h_e \star \tilde{h}_e \star h_{1o} + h_{1o} \star \tilde{h}_o \star h_o] \\ &= [h_e \star \tilde{h}_e + h_o \star \tilde{h}_o] \star h_{1o} \\ &= h_{1o}. \end{aligned}$$

Similarly by convolving both sides of [\[link\]](#) by \tilde{h}_o we get

Equation:

$$\mp \tilde{h}_o \star \delta_k = h_{1e}.$$

Combining [\[link\]](#), and [\[link\]](#), gives the result

Equation:

$$h_1(n) = \pm(-1)^n h(-n+1-2k).$$

Proof 11 We show the integral of the wavelet is zero by integrating both sides of ([\[link\]](#)) gives

Equation:

$$\int \psi(t) dt = \sum_n h_1(n) \int \sqrt{2} \varphi(2t-n) dt$$

But the integral on the right hand side is A_0 , usually normalized to one and from [\[link\]](#), or [\[link\]](#), and [\[link\]](#), we know that

Equation:

$$\sum_n h_1(n) = 0$$

and, therefore, from [\[link\]](#), the integral of the wavelet is zero.

The fact that multiplying in the time domain by $(-1)^n$ is equivalent to shifting in the frequency domain by π gives $H_1(\omega) = H(\omega + \pi)$.